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Categorical completeness results for the simply-typed lambda-calculus

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Abstract. We investigate, in a categorical setting, some completeness properties of beta-eta conversion between closed terms of the simply-typed lambda calculus. A cartesian-closed category is said to be *complete* if, for any two unconvertible terms, there is some interpretation of the calculus in the category that distinguishes them. It is said to have a *complete interpretation* if there is some interpretation that equates only interconvertible terms. We give simple necessary and sufficient conditions on the category for each of the two forms of completeness to hold. The classic completeness results of, e.g., Friedman and Plotkin are immediate consequences. As another application, we derive a syntactic theorem of Statman characterizing beta-eta conversion as a maximum consistent congruence relation satisfying a property known as typical ambiguity.

1 Introduction

In 1970 Friedman proved that beta-eta conversion is complete for deriving all equalities between the (simply-typed) lambda-definable functionals in the category **Set** [5]. (Incidentally, this result was independently discovered by Plotkin [10], published in [11].) However, in computer science one is often interested in interpretations in other cartesian closed categories (such as the category of complete partial orders and continuous functions). It is natural to ask whether similar completeness results also hold in such cases. For the category of complete partial orders, Plotkin was able to extend Friedman's argument and show that completeness does indeed still hold (see [9, Theorem 5.2.28]). More recently, Berger and Schwichtenberg used different techniques to show that completeness holds relative to any model capable of faithfully representing certain basic operations on syntax [3].

In this paper we investigate such completeness questions in a categorical setting. As is well known, cartesian-closed categories (CCCs) provide a general notion of model for the simply-typed lambda calculus. We ask under what conditions on a CCC, $\mathcal{C}$, does beta-eta conversion derive all equalities between terms which are true in $\mathcal{C}$. Actually, this question is not yet well defined, as different interpretations of base types in $\mathcal{C}$ might induce different equalities. Thus there are two natural strengths of completeness. The weaker form holds when beta-eta conversion derives all those equalities between terms which are true under all interpretations in $\mathcal{C}$. The stronger form holds when there is a single interpretation that equates only terms that are beta-eta convertible. In this paper we give necessary and sufficient conditions on $\mathcal{C}$ for each of the forms of completeness to hold (Theorems 1 and 2). The conditions turn out to be simple ones that are easily checked in particular cases. Moreover, they show the failure of completeness to be the exception rather than the rule.

As an application, we use Theorem 1 to obtain Statman's [16] characterization of beta-eta convertibility as a maximally consistent congruence relation satisfying typical ambiguity (Theorem 3). Indeed, as will be seen, our work is closely related to, and also heavily dependent upon, some fundamental syntactic work of Statman. We shall discuss this dependency further in Section 7.

2 Preliminaries

In order to have a tight connection between the lambda-calculus and cartesian-closed categories we work with a calculus with finite product types. We use $\alpha, \beta, \ldots$ to range over a non-empty set of base types, X, containing a distinguished base type, 0. We use $\sigma, \tau, \ldots$ to range over types which comprise: base types, function types $\sigma \to \tau$, (binary) product types $\sigma \times \tau$, and a unit type $\mathbf{1}$. We work with explicitly typed variables $x^\sigma, y^\tau, \ldots$ although we often omit type labels for convenience. We use $U, V, \ldots$ to range over open terms which are given by the grammar:

$$U ::= x^\sigma \mid \lambda x^\sigma . U \mid U(V) \mid \langle U, V \rangle \mid \pi_1(U) \mid \pi_2(U) \mid *$$

(where $\langle U, V \rangle$ and $\pi_i(U)$ are pairing and projection for product types and $*$ is the canonical element of $\mathbf{1}$) subject to the usual typing constraints. Each term has a unique type and we write U^σ to mean that the type of U is σ. We use $L, M, N, \ldots$ to range over *closed* terms. We write Λ_X for the set of *closed* terms. We write $\Lambda_X^{\to}$ for those terms in Λ_X that are terms of the usual pure functionally typed lambda-calculus (i.e. those terms all of whose subterms have types built from X using $\to$). We adopt standard conventions such as associating $\to$ to the right and application to the left. We also use evident notation for products of arbitrary finite arity, their tuples and projections.

We assume that the reader is acquainted with the rules for beta-eta convertibility, $=_{\beta\eta}$, between terms of identical type (see, e.g, [1, 4, 7]). Two classes of terms, the *neutral terms* and the *long-$\beta\eta$ normal forms*, are defined by mutual induction. A term is *neutral* if it has one of the following forms: x^σ; or $U(V)$ where U is neutral and V is in long-$\beta\eta$ normal form; or $\pi_i(U)$ where U is neutral. A term is in *long-$\beta\eta$ normal form* if it has one of the following forms: U^α where U is neutral (note the restriction to a base type); or $\lambda x^\sigma . U$ where U is in long-$\beta\eta$ normal form; or $\langle U, V \rangle$ where U and V are both in long-$\beta\eta$ normal form; or $*$. The important fact about long-$\beta\eta$ normal forms is that, for every term U, there is a unique long-$\beta\eta$ normal form, $\beta\eta(U)$, such that $U =_{\beta\eta} \beta\eta(U)$ (see [1, 4, 7]). By this characterization it is clear that $=_{\beta\eta}$ between terms in Λ_X is conservative over the usual beta-eta convertibility between terms in $\Lambda_X^{\to}$.

Let $\mathcal{C}$ be a cartesian-closed category with distinguished: terminal object, $\mathbf{1}$; binary products, $A \times B$; and exponentials, B^A. (We do not assume that $\mathcal{C}$ has all finite limits.) An interpretation of the calculus in $\mathcal{C}$ is determined by a function $[\![\cdot]\!]$ from X to objects of $\mathcal{C}$. This extends (using the CCC structure of $\mathcal{C}$) to interpret arbitrary types σ as objects $[\![\sigma]\!]$ of $\mathcal{C}$. Then a closed term M^σ is interpreted as a morphism $[\![M]\!] \in \mathcal{C}(\mathbf{1}, [\![\sigma]\!])$. (The interpretation is defined using a more general interpretation of open terms, U^σ, as morphisms from objects interpreting the context of free variables in U to $[\![\sigma]\!]$.) We write $\Lambda_X \to \mathcal{C}$ for the class of all interpretations of the calculus in $\mathcal{C}$. The soundness of beta-eta conversion in CCCs says that $M =_{\beta\eta} N$ implies that, for all $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$, it holds that $[\![M]\!] = [\![N]\!]$. We shall be interested in when the converse implication holds, and related questions.

Before considering such completeness questions we consider the categorical formulation of what an interpretation of the lambda-calculus in $\mathcal{C}$ is (see [8]). This formulation is in terms of cartesian-closed functors (CC-functors), which are those functors between CCCs that preserve the cartesian-closed structure "on the nose".[1] Let $\mathcal{F}_X$ be the free cartesian-closed category generated by the set of objects X. To give a concrete description, $\mathcal{F}_X$ is the category whose objects are types and whose morphisms from σ to τ are the closed long-$\beta\eta$ normal forms of type $\sigma \to \tau$. The identities and composition are obtained as the long-$\beta\eta$ normal forms of the evident lambda-terms. The freeness of $\mathcal{F}_X$ means that any function $[\![\cdot]\!]$ from X to objects of $\mathcal{C}$ extends to a unique CC-functor, F, from $\mathcal{F}_X$ to $\mathcal{C}$, where "extends" means that $F(\alpha) = [\![\alpha]\!]$. Further, if we write $[\![\cdot]\!]$ for the interpretation of the lambda-calculus induced by the function on X, it holds that, for all M^σ, $[\![M]\!] = F(\beta\eta(\lambda x^1 .\, M)) \in \mathcal{C}(\mathbf{1}, [\![\sigma]\!])$ and, for all long-$\beta\eta$ normal forms $M^{\sigma\to\tau}$ that $F(M) \in \mathcal{C}([\![\sigma]\!], [\![\tau]\!])$ is the evident exponential transpose of $[\![M]\!] \in \mathcal{C}(\mathbf{1}, [\![\tau]\!]^{[\![\sigma]\!]})$. Thus interpretations of the lambda-calculus in $\mathcal{C}$ are essentially equivalent to CC-functors from $\mathcal{F}_X$ to $\mathcal{C}$.

3 The Completeness Theorems

We now define the two forms of completeness we shall be investigating. First the weaker notion, which is the direct converse to the soundness statement above. We say that $\mathcal{C}$ is *complete (for $=_{\beta\eta}$)*[2] if, for all M^σ, N^σ,

$$M =_{\beta\eta} N \text{ iff for all } [\![\cdot]\!] : \Lambda_X \to \mathcal{C}, \quad [\![M]\!] = [\![N]\!].$$

This concept has a natural categorical formulation. Recall that a class of functors from a category $\mathcal{A}$ to a category $\mathcal{B}$ is *collectively faithful* if, for all $A \xrightarrow{f} B$ and $A \xrightarrow{g} B$ in $\mathcal{A}$, whenever it holds that $F(f) = F(g)$ for all functors F in the

[1] The whole discussion here could easily be generalized to deal with functors preserving the structure up to isomorphism. Such functors are categorically more natural, but for our purposes the simpler "on the nose" functors suffice.

[2] It would perhaps be preferable to say that $=_{\beta\eta}$ is complete for $\mathcal{C}$, however this is not so easily shortened.

class, then $f = g$. Thus, using the equivalence between interpretations and CC-functors, $\mathcal{C}$ is complete if and only if the class of CC-functors from $\mathcal{F}_X$ to $\mathcal{C}$ is collectively faithful.

For the stronger notion we require completeness relative to a single interpretation rather than the class of all interpretations. We say that an interpretation $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$ is *complete (for $=_{\beta\eta}$)* if, for all M^σ, N^σ,

$$M =_{\beta\eta} N \text{ iff } [\![M]\!] = [\![N]\!].$$

We say that $\mathcal{C}$ *has a complete interpretation (for $=_{\beta\eta}$)* if there exists a complete interpretation $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$. Again these concepts have natural categorical reformulations. An interpretation is complete if and only if the corresponding CC-functor from $\mathcal{F}_X$ to $\mathcal{C}$ is faithful. Similarly, $\mathcal{C}$ has a complete interpretation if and only if there exists a faithful CC-functor from $\mathcal{F}_X$ to $\mathcal{C}$.

In this paper we characterize the conditions under which $\mathcal{C}$ is complete (Theorem 1) and under which $\mathcal{C}$ has a complete interpretation (Theorem 2). It is also interesting to consider the question of characterizing when a given interpretation $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$ is complete. This problem is of a different nature as it no longer concerns a property intrinsic to the category $\mathcal{C}$. In the case that $X = \{0\}$, such a characterization (essentially due to Statman) will be obtained in Section 4 (Corollary 4). We do not have such a result for arbitrary X. Some of the problems in obtaining one will be considered in Section 6.

Before giving the characterizations, we consider some motivating examples. First, the category **Set** has a complete interpretation. Indeed any interpretation mapping each base type to an infinite set is complete. This result is proved explicitly in [4], but it is closely related to Friedman's famous completeness theorem [5]. (There is a detailed discussion of the differences in [4].) It is clear then that **Set** is complete, as in general the existence of a complete interpretation implies completeness. The converse is not true. An example that is complete but which has no complete interpretation is the category of finite sets, **FinSet**. The completeness of **FinSet** is proved explicitly in [13], but it is closely related to Theorem 2 of [15] (a result originally due to Plotkin [10]), which is basically a finite model property for beta-eta conversion. The non-existence of a complete interpretation in **FinSet** was essentially observed by Friedman [5]. The reason is simply that there exist types with an infinite number of equivalence classes of closed terms modulo $=_{\beta\eta}$, for example $(0 \to 0) \to 0 \to 0$. Lastly, there do indeed exist cartesian-closed categories that are not complete. Recall that a *preorder* is a category with at most one morphism in each hom-set. It is obvious that any cartesian-closed preorder (for example, any Heyting algebra) is not complete.

The first characterization says that the preorder observation above is the only obstacle to completeness.

Theorem 1 *$\mathcal{C}$ is complete if and only if it is not a preorder.*

So, perhaps surprisingly, completeness turns out to be merely a question of the non-triviality of the hom-sets of $\mathcal{C}$.

We have seen that completeness is determined by the simple cardinality condition that there exists a hom-set with cardinality ≥ 2. Given that the counterexample to a complete interpretation in **FinSet** is also via a cardinality argument, one might wonder whether $\mathcal{C}$ has a complete interpretation if and only if it has an infinite hom-set. This, however, is not the case. For a counterexample take the full subcategory of the co-Kleisli category of the $\omega \times -$ comonad on **Set** determined by those objects that are the image of finite sets under the inclusion from **Set** to the co-Kleisli category. We call this category $\mathbf{FinSet}_{\omega\times -}$. (More concretely, $\mathbf{FinSet}_{\omega\times -}$ has finite sets for objects, and the morphisms from X to Y are those functions from $\omega \times X$ to $\omega \times Y$ that preserve the first component of pairs.) Theorem 2 below gives an elementary way of checking that there is indeed no complete interpretation in $\mathbf{FinSet}_{\omega\times -}$. A more abstract reason for this failure is that any CC-functor from $\mathcal{F}_X$ to $\mathbf{FinSet}_{\omega\times -}$ necessarily factors through the inclusion from **FinSet**. This can be proved using the universal property of the co-Kleisli category as a polynomial category (see [8]) together with the initiality of $\mathcal{F}_X$. We omit the argument.

Nevertheless, a closely related condition does succeed in characterizing the existence of a complete interpretation. We say that an endomorphism $A \xrightarrow{a} A$ is *non-repeating* if all its iterates are distinct (i.e. if $a^h = a^k$ implies $h = k$).

Theorem 2 *$\mathcal{C}$ has a complete interpretation if and only if it contains a non-repeating endomorphism.*

Note that it is not apparent from the definitions of the two forms of completeness that they are independent of the choice of X. Theorems 1 and 2 show this to be the case.

4 Proofs of Theorems 1 and 2

We shall prove Theorem 2 first and then derive Theorem 1 as a consequence. Throughout the proofs we move freely between categorical formulations in terms of CC-functors and syntactic formulations in terms of interpretations. We also move freely between the interpretations of terms as morphisms in $\mathcal{C}(\mathbf{1}, [\![\tau]\!]^{[\![\sigma]\!]})$ and their exponential transposes as morphisms in $\mathcal{C}([\![\sigma]\!], [\![\tau]\!])$.

For the left-to-right implication of Theorem 2, suppose that $[\![\cdot]\!]$ is a complete interpretation. Not surprisingly, a non-repeating endomorphism is given by a successor function on the interpretation of the Church numerals. Specifically, the endomorphism is:[3]

$$[\![(0 \to 0) \to 0 \to 0]\!] \xrightarrow{[\![\lambda x^{(0\to 0)\to 0\to 0}.\, \lambda y^{0\to 0}.\, \lambda z^{0}.(x(y)(y(z)))]\!]} [\![(0 \to 0) \to 0 \to 0]\!]$$

(making use of an exponential transpose as discussed above). It is non-repeating because if its n-th iterate is composed with

$$\mathbf{1} \xrightarrow{[\![\lambda y^{0\to 0}.\, \lambda z^{0}.z]\!]} [\![(0 \to 0) \to 0 \to 0]\!]$$

[3] This and other "diagrams" were prepared using Paul Taylor's Latex diagram macros package.

then one obtains:

$$\mathbf{1} \xrightarrow{[\![\lambda y^{0\to 0}.\,\lambda z^0.y^n(z)]\!]} [\![(0\to 0)\to 0\to 0]\!],$$

and, by completeness, it is clear that the latter differs for distinct values of n. Incidentally, here we have shown that it is a necessary condition for $[\![\cdot]\!]$ to be complete that the above endomorphism is non-repeating. In Section 6 we show that this is not in general a sufficient condition, even for interpretations of $\Lambda_{\{0\}}$.

For the converse implication, given a non-repeating endomorphism in $\mathcal{C}$, we must construct a faithful CC-functor from $\mathcal{F}_X$ to $\mathcal{C}$.

Proposition 1 *There is a faithful CC-functor from $\mathcal{F}_X$ to $\mathcal{F}_{\{0\}}$.*

Proof. The CC-functor is that determined by the unique function from X to $\{0\}$. This maps any X-type, σ, to the $\{0\}$-type, $\overline{\sigma}$, obtained by replacing every base type α with 0. For any $M^\sigma \in \Lambda_X$ define $\overline{M}$ to be the $\Lambda_{\{0\}}$-term obtained by replacing every variable x^σ in M with $x^{\overline{\sigma}}$. Clearly $\overline{M}$ has type $\overline{\sigma}$. For faithfulness it is enough to show that, for any two distinct long-$\beta\eta$ normal forms $M^\sigma, N^\sigma \in \Lambda_X$, it holds that $\overline{M}$ and $\overline{N}$ are distinct long-$\beta\eta$ normal forms in $\Lambda_{\{0\}}$. This is done by a straightforward induction on the structure of long-$\beta\eta$ normal forms. ⊠

Thus it remains to find a faithful CC-functor from $\mathcal{F}_{\{0\}}$ to $\mathcal{C}$. For this we appeal to a deep syntactic result about the (pure functional) simply-typed lambda-calculus due to Statman [15, Theorem 3]. Define $\top$ to be the type $(0\to 0\to 0)\to 0\to 0$.

Proposition 2 (Statman) *For all $M^\sigma, N^\sigma \in \Lambda^{\to}_{\{0\}}$, it holds that $M =_{\beta\eta} N$ if and only if, for all $L^{\sigma\to\top}$, $L(M) =_{\beta\eta} L(N)$.*

A detailed proof can be found in [12]. Incidentally, in [14, Proposition 1], Statman shows that, for each σ, there exists $L^{\sigma\to\top}$, such that $M =_{\beta\eta} N$ if and only if $L(M) =_{\beta\eta} L(N)$, but we do not need this stronger result here.

Proposition 3 *For all $M^\sigma, N^\sigma \in \Lambda_{\{0\}}$, it holds that $M =_{\beta\eta} N$ if and only if, for all $L^{\sigma\to\top}$, $L(M) =_{\beta\eta} L(N)$.*

Proof. Left-to-right is trivial. For the converse suppose that $M^\sigma \neq_{\beta\eta} N^\sigma$. It is easily shown that σ is isomorphic (in $\mathcal{F}_{\{0\}}$) to a finite product $\sigma_1 \times \ldots \times \sigma_n$ (where $n \geq 0$) of types σ_i built from 0 using $\to$. We write σ' for this product type and $I^{\sigma\to\sigma'}$ for the lambda-term giving (one half of) the isomorphism. Clearly $I(M) \neq_{\beta\eta} I(N)$, so $\langle \pi_1(I(M)), \ldots, \pi_n(I(M))\rangle \neq_{\beta\eta} \langle \pi_1(I(N)), \ldots, \pi_n(I(N))\rangle$. Therefore there is some i for which $\pi_i(I(M)) \neq_{\beta\eta} \pi_i(I(N))$. So $\beta\eta(\pi_i(I(M))) \neq \beta\eta(\pi_i(I(N)))$. But these terms are both normal forms of type σ_i, and hence they are terms of $\Lambda^{\to}_{\{0\}}$ (because all subterms of a normal form have subtypes of its type). So, by Proposition 2, there exists $L^{\sigma_i\to\top}$ such that $L(\pi_i(I(M))) \neq_{\beta\eta} L(\pi_i(I(N)))$. But then $\lambda x^\sigma.\, L(\pi_i(I(x)))$ is the term of type $\sigma \to \top$ that we are trying to find. ⊠

Corollary 4 *An interpretation, $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathcal{C}$, is complete if and only if for all $M^\top$, $N^\top$ it holds that $[\![M]\!] = [\![N]\!]$ implies $M =_{\beta\eta} N$.*

Proof. Left-to-right is trivial. For the converse, suppose that for all $M^\top$, $N^\top$ it holds that $[\![M]\!] = [\![N]\!]$ implies $M =_{\beta\eta} N$. Suppose that $[\![M^\sigma]\!] = [\![N^\sigma]\!]$. By the "compositionality" of $[\![\cdot]\!]$ we have, for all $L^{\sigma\to\top}$, that $[\![L(M)]\!] = [\![L(N)]\!]$. Whence, by the assumption, for all $L^{\sigma\to\top}$, we have $L(M) =_{\beta\eta} L(N)$. So, by Proposition 3, $M =_{\beta\eta} N$. Thus $[\![\cdot]\!]$ is indeed complete. ⊠

The corollary gives a necessary and sufficient condition for an interpretation of $\Lambda_{\{0\}}$ in $\mathcal{C}$ to be complete. We use this to obtain a useful sufficient condition. A *very weak natural number object* in $\mathcal{C}$ is an object B together with morphisms:

$$\mathbf{1} \xrightarrow{\overline{0}} B \xrightarrow{s} B \underset{\times}{\overset{+}{\leftleftarrows}} B \times B$$

such that, for all m, n, it holds that $\overline{m+n} = + \circ \langle \overline{m}, \overline{n} \rangle$ and $\overline{m \times n} = \times \circ \langle \overline{m}, \overline{n} \rangle$, where we write $\overline{n}$ for the "numeral" morphism $s^n \circ \overline{0}$.[4] A very weak natural number object is said to be *faithful* if all the numerals are distinct (i.e. if $\overline{m} = \overline{n}$ implies $m = n$).

The next lemma generalizes the completeness theorem that appears in Berger and Schwichtenberg [3] (although they work in a non-categorical setting).

Lemma 5 *An interpretation, $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathcal{C}$, is complete if $[\![0]\!]$ is a faithful very weak natural number object.*

Proof. Let B be $[\![0]\!]$. Let ϕ be the binary function on natural numbers defined by $\phi(m, n) = (m+n)^2 + m + 1$. By simple composition using the very weak natural number morphisms, there is a morphism $B \times B \xrightarrow{\overline{\phi}} B$ such that $\overline{\phi} \circ \langle \overline{m}, \overline{n} \rangle = \overline{\phi(m, n)}$. We shall use this to show that the condition of Corollary 4 is satisfied.

First, it is routine to check that the closed long-$\beta\eta$ normal forms of type $\top$ have the form $\lambda p^{0\to 0\to 0}.\, \lambda l^0.\, t$ where t is given by the grammar:

$$t ::= l \quad | \quad p(t_1)(t_2).$$

Now we define inductively a numerically valued function, $(\cdot)^*$, on the set of such t by:

$$l^* = 0,$$
$$(p(t_1)(t_2))^* = \phi(t_1^*, t_2^*).$$

It is easily seen that $t_1^* = t_2^*$ implies t_1 and t_2 are identical (as ϕ is an injective function from $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}^+$).

[4] Note that there is no requirement that $+$ and $\times$ satisfy any of the usual algebraic identities.

Now, for any t, we have a morphism $[\![\lambda p.\,\lambda l.\,t]\!] \in \mathcal{C}(\mathbf{1}, \top)$ and we note the evident corresponding $B^{(B\times B)} \times B \xrightarrow{\tilde{t}} B$. We also note the exponential transpose $\mathbf{1} \xrightarrow{\tilde{\phi}} B^{(B\times B)}$ of $\overline{\phi}$. It is easily checked that the composite:

$$\mathbf{1} \xrightarrow{\langle \tilde{\phi}, \overline{0} \rangle} B^{(B\times B)} \times B \xrightarrow{\tilde{t}} B$$

is equal to $\overline{t^*}$. So if $[\![\lambda p.\,\lambda l.\,t_1]\!] = [\![\lambda p.\,\lambda l.\,t_2]\!]$ then $\overline{t_1^*} = \overline{t_2^*}$ and thus t_1 and t_2 are identical (as the very weak natural number object is faithful).

To complete the proof, suppose that $[\![M^\top]\!] = [\![N^\top]\!]$. Suppose that $\beta\eta(M) = \lambda p.\,\lambda l.\,t_1$ and $\beta\eta(N) = \lambda p.\,\lambda l.\,t_2$. Then, by the above, t_1 and t_2 are identical so $M =_{\beta\eta} N$. Thus the condition of Corollary 4 is indeed satisfied. ⊠

Note that the condition of the lemma is not necessary for $[\![\cdot]\!]$ to be complete. It fails, for example, for the evidently complete "identity" interpretation of $\Lambda_{\{0\}}$ in $\mathcal{F}_{\{0\}}$ where there is no morphism from $\mathbf{1}$ to $[\![0]\!]$ (and the only endomorphism on $[\![0]\!]$ is the identity).

Let $A \xrightarrow{a} A$ be a non-repeating endomorphism in $\mathcal{C}$. Let B be the object $(A^A)^{(A^A)}$. We use the internal lambda-calculus of $\mathcal{C}$ to define arrows:

$$\begin{array}{rcl}
\overline{0} & = & \mathbf{1} \xrightarrow{\lambda f^{A^A}.\lambda a^A.a} B \\
s & = & B \xrightarrow{b \;\mapsto\; \lambda f^{A^A}.\lambda a^A.b(f)(f(a))} B \\
+ & = & B \times B \xrightarrow{\langle b, b' \rangle \;\mapsto\; \lambda f^{A^A}.\lambda a^A.b(f)(b'(f)(a))} B \\
\times & = & B \times B \xrightarrow{\langle b, b' \rangle \;\mapsto\; \lambda f^{A^A}.\lambda a^A.b(b'(f))(a)} B
\end{array}$$

making use of standard encodings of successor, addition and multiplication on Church numerals. It is clear that these morphisms show B to be a very weak natural number object. To see that it is faithful note that, by exponential transpose, each $\overline{n}$ gives a morphism $A^A \xrightarrow{\tilde{n}} A^A$ and a gives a morphism $\mathbf{1} \xrightarrow{\tilde{a}} A^A$. It is easily seen that the exponential transpose of the composite $\tilde{n} \circ \tilde{a}$ is $A \xrightarrow{a^n} A$. Thus the numerals must all be distinct as otherwise would contradict a being a non-repeating endomorphism.

It now follows from Lemma 5 that the interpretation $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathcal{C}$ determined by setting $[\![0]\!] = B$ is complete. Together with Proposition 1, this completes the proof of Theorem 2.

We now turn to Theorem 1. The left-to-right implication is trivial. For the converse, suppose that $\mathcal{C}$ is not a preorder. We shall show that there is a faithful CC-functor, F, from $\mathcal{F}_X$ to $\mathcal{C}^\omega$ (the countably infinite power of $\mathcal{C}$), which is indeed a CCC. Given such an F, a collectively faithful set of CC-functors from $\mathcal{F}_X$ to $\mathcal{C}$ is $\{\pi_i \circ F \mid i \in \omega\}$ where π_i is is the i-th projection from $\mathcal{C}^\omega$ to $\mathcal{C}$ (it is easily checked that the projections are CC-functors), from which it is clear that the class of all CC-functors is collectively faithful.

To obtain F we use Theorem 2, by which it suffices to find a non-repeating endomorphism in $\mathcal{C}^\omega$. As $\mathcal{C}$ is not a preorder, suppose that f and g are two distinct morphisms in $\mathcal{C}(A, B)$. For $n \geq 1$ define $B_n = B^{B^n}$ where B^n is the n-fold product of B with itself. For $i \in \{0, \ldots, n-1\}$ define:

$$\begin{aligned} \overline{i_n} &= \mathbf{1} \xrightarrow{\lambda c^{B^n}.\, \pi_i(c)} B_n \\ s_n &= B_n \xrightarrow{d \;\mapsto\; \lambda c^{B^n}.\, d(\langle \pi_2(c), \ldots, \pi_n(c), \pi_1(c) \rangle)} B_n. \end{aligned}$$

Clearly $s_n \circ \overline{i_n} = \overline{j_n}$ where j is $i+1$ modulo n. We now show that $\overline{0_n}, \ldots, \overline{(n-1)_n}$ are all distinct. Let $B^n \xrightarrow{\tilde{i_n}} B$ be the exponential transpose of i_n. It is clear that the composite:

$$A \xrightarrow{\langle \overbrace{f, \ldots, f}^{j}, g, \ldots, g \rangle} B^n \xrightarrow{\tilde{i_n}} B$$

is equal to f if $i \leq j$ and is equal to g otherwise. This shows that $j > i$ implies $\overline{i_n} \neq \overline{j_n}$ (as $f \neq g$), so $\overline{0_n}, \ldots, \overline{(n-1)_n}$ are indeed all distinct. It is now clear that

$$(B_1, B_2, \ldots) \xrightarrow{(s_1, s_2, \ldots)} (B_1, B_2, \ldots)$$

is a non-repeating endomorphism in $\mathcal{C}^\omega$, as required.

5 Typical Ambiguity

In this section we apply Theorem 1 to obtain a syntactic characterization of $=_{\beta\eta}$ as, in a sense to be defined below, a maximally consistent congruence relation satisfying typical ambiguity (Theorem 3). For the calculus $\Lambda^{\rightarrow}_{\{0\}}$, this result is originally due to Statman [16]. Although the theorem for Λ_X is easily derived from Statman's result for $\Lambda^{\rightarrow}_{\{0\}}$, it is an interesting application of our completeness results to obtain it instead as a consequence of Theorem 1. As a matter of fact, we shall also see that one can turn the tables and derive Theorem 1 from Theorem 3. Thus, in some sense, Theorem 1 is a semantic counterpart to the syntactic Theorem 3.

First we introduce the necessary notation to state Theorem 3. Given a function ϕ from X to types, we write $\sigma[\phi]$ for the type obtained by simultaneously replacing each occurrence of a base type α in σ with $\phi(\alpha)$. Similarly, we write $M[\phi]$ for the term obtained by replacing all variables x^σ in M with $x^{\sigma[\phi]}$. If M has type σ then $M[\phi]$ has type $\sigma[\phi]$. Such a substitution of types clearly corresponds to a CC-functor from $\mathcal{F}_X$ to itself.

Let $\sim$ be a well-typed equivalence relation on Λ_X (i.e. one for which $M \sim N$ implies M and N are of identical type) such that $M =_{\beta\eta} N$ implies $M \sim N$. We say that $\sim$ is a *congruence* if $M^\sigma \sim N^\sigma$ implies that, for all $L^{\sigma \rightarrow \tau}$, $L(M) \sim L(N)$ (the other properties of a congruence relation follows from this because $\sim$ contains $=_{\beta\eta}$). We say that $\sim$ is *consistent* if, for some σ, there exist two terms, M^σ and N^σ, such that $M \not\sim N$. We say that $\sim$ satisfies *typical ambiguity* if, for all type-valued functions, ϕ, on X, it holds that $M \sim N$ implies $M[\phi] \sim N[\phi]$.

Theorem 3 *If $\sim$ is a consistent congruence relation containing $=_{\beta\eta}$ and $\sim$ satisfies typical ambiguity then $M \sim N$ if and only if $M =_{\beta\eta} N$.*

To prove the theorem, suppose that $\sim$ satisfies the assumptions. We construct a category $\mathcal{F}_X/\!\sim$ as follows. The objects of $\mathcal{F}_X/\!\sim$ are types. The morphisms from σ to τ are the equivalence classes of the set of closed terms of type $\sigma \to \tau$ modulo $\sim$, and we write $[M]$ for the equivalence class of M. The identities and composition are evident. It is easily checked that $\mathcal{F}_X/\!\sim$ is a CCC, using the fact that $\sim$ extends $=_{\beta\eta}$ and the congruence property of $\sim$. Further, by the consistency property, $\mathcal{F}_X/\!\sim$ is not a preorder.

Lemma 6 *Given any $[\![\cdot]\!] : \Lambda_X \to \mathcal{F}_X/\!\sim$, define ϕ from X to types by $\phi(\alpha) = [\![\alpha]\!]$. Then $[\![M]\!] = [\lambda x^1.\, M[\phi]]$ in $\mathcal{F}_X/\!\sim(\mathbf{1}, \sigma[\phi])$.*

This is proved by induction on the structure of M. The induction, which involves going through interpretations of open terms, is routine.

Now suppose that $M \sim N$. Let $[\![\cdot]\!]$ be any interpretation in $\mathcal{F}_X/\!\sim$, and define ϕ as above. By typical ambiguity, $M[\phi] \sim N[\phi]$. Whence, by the congruence property, $\lambda x^1.\, M[\phi] \sim \lambda x^1.\, N[\phi]$. So it follows from the lemma that $[\![M]\!] = [\![N]\!]$. We have shown that, for any $[\![\cdot]\!]$, we have that $[\![M]\!] = [\![N]\!]$. Thus Theorem 1 implies that $M =_{\beta\eta} N$. This proves Theorem 3.

As commented above, one can also derive Theorem 1 from Theorem 3. To this end, suppose that $\mathcal{C}$ is not a preorder. Define a well-typed equivalence relation, $\sim$, by:

$$M \sim N \text{ iff for all } [\![\cdot]\!] : \Lambda_X \to \mathcal{C}, \quad [\![M]\!] = [\![N]\!].$$

By the soundness of $=_{\beta\eta}$, we have that $\sim$ contains $=_{\beta\eta}$. Theorem 1 says that $M =_{\beta\eta} N$ if and only if $M \sim N$. To show this we need only verify that $\sim$ satisfies the conditions of Theorem 3. The congruence property is straightforward (it holds because of the "compositionality" of $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$). Consistency follows from $\mathcal{C}$ not being a preorder, as it is easy to find an interpretation such that $[\![\lambda x^0.\, \lambda y^0.\, x]\!] \neq [\![\lambda x^0.\, \lambda y^0.\, y]\!]$. It remains to show typical ambiguity. First we note the lemma below, which is proved by a straightforward induction on the structure of M (again involving interpretations of open terms).

Lemma 7 *Given any ϕ from X to types and interpretation $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$, let $[\![\cdot]\!]'$ be the interpretation determined by $[\![\alpha]\!]' = [\![\phi(\alpha)]\!]$. Then $[\![M[\phi]]\!] = [\![M]\!]'$.*

Suppose that $M \sim N$. Let $[\![\cdot]\!]$ be any interpretation. By the lemma, we have that $[\![M[\phi]]\!] = [\![M]\!]'$ and $[\![N[\phi]]\!] = [\![N]\!]'$. Now $M \sim N$, so by the definition of $\sim$ we have that $[\![M]\!]' = [\![N]\!]'$. Therefore $[\![M[\phi]]\!] = [\![N[\phi]]\!]$. So $M[\phi] \sim N[\phi]$, and $\sim$ does indeed satisfy typical ambiguity.

The derivation of Theorem 1 from Theorem 3, gives a proof of Theorem 1 not involving Theorem 2. However, Statman's proof of Theorem 3 (for $\Lambda^{\to}_{\{0\}}$) also relies on the reduction of $=_{\beta\eta}$ to the single type $\top$ (Proposition 2), on which our proof of Theorem 2 was based. It is an interesting fact that an alternative direct proof of Theorem 3 is possible using a typed version of the Böhm-out technique [2, Ch. 10]. The details are beyond the scope of this paper.

6 Complete Interpretations

In this section we consider the problem of obtaining a characterization of when a given interpretation is complete. Corollary 4 already characterizes when an interpretation $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathcal{C}$ is complete. We consider whether this characterization can be improved in a natural way. We also consider whether it generalizes to interpretations of Λ_X for an arbitrary X. Although the results we obtain are negative, they do illustrate well some of the more delicate aspects of the completeness questions.

One natural question is whether Corollary 4 can be improved by simplifying the type of M and N from $\top$ to $(0 \to 0) \to 0 \to 0$. Below, we use logical relations to construct a model answering this questions in the negative. This negative answer justifies the comment made at the end of our proof of the left-to-right implication of Theorem 2. In general it is an insufficient condition for an interpretation $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathcal{C}$ to be complete that the interpretation of the successor function on Church numerals be a non-repeating endomorphism.

The category $\mathbf{R_3}$ is defined as follows. Its objects A are pairs $(|A|, R_A)$ where $|A|$ is a set and R_A is a ternary relation on $|A|$ such that $R_A(a,a,a)$ for all $a \in |A|$. The morphisms from A to B are those functions $f : |A| \to |B|$ such that, for all $a_1, a_2, a_3 \in |A|$, it holds that $R_A(a_1,a_2,a_3)$ implies $R_B(f(a_1), f(a_2), f(a_3))$. This category is cartesian closed with: $|\mathbf{1}| = \{\emptyset\}$ where $R_1(\emptyset,\emptyset,\emptyset)$ holds; and $|A \times B| = |A| \times |B|$ with $R_{A\times B}(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \langle a_3, b_3\rangle)$ if and only if $R_A(a_1,a_2,a_3)$ and $R_B(b_1,b_2,b_3)$; and $|B^A| = \mathbf{R_3}(A,B)$ with $R_{B^A}(f_1,f_2,f_3)$ if and only if, for all $a_1,a_2,a_3 \in |A|$, it holds that $R_A(a_1,a_2,a_3)$ implies $R_B(f_1(a_1), f_2(a_2), f_3(a_3))$. The details are easily checked. Let A be the object of $\mathbf{R_3}$ defined by $|A| = \omega$ and $R_A(l,m,n)$ if and only if either $l = m = n$ or $l+1 = m = n-1$.

Define an interpretation $[\![\cdot]\!] : \Lambda_{\{0\}} \to \mathbf{R_3}$ by setting $[\![0]\!] = A$. We claim that, for all $M^{(0\to 0)\to 0\to 0}, N^{(0\to 0)\to 0\to 0}$, it holds that $[\![M]\!] = [\![N]\!]$ implies $M =_{\beta\eta} N$. Note that the closed long-$\beta\eta$ normal forms of $(0 \to 0) \to 0 \to 0$ have the form $\lambda s.\, \lambda z.\, s^n(z)$ for $n \geq 0$. As the function $n \mapsto n+1$ is in $\mathbf{R_3}(A,A)$, and hence in $|A^A|$, it is easily seen that any two distinct long-$\beta\eta$ normal forms of $(0 \to 0) \to 0 \to 0$ get interpreted as different functionals in $[\![(0 \to 0) \to 0 \to 0]\!]$. The claim follows.

Despite completeness for the type $(0 \to 0) \to 0 \to 0$, it turns out that $[\![\cdot]\!]$ is not complete. By Corollary 4, we know that the incompleteness must already arise for terms of type $\top$.

Lemma 8 *A function $f : A \times A \to A$ is in $\mathbf{R_3}(A \times A, A)$ if and only if, for some $k \geq 0$, it holds that f is one of: $\langle m,n\rangle \mapsto k$; or $\langle m,n\rangle \mapsto m+k$; or $\langle m,n\rangle \mapsto n+k$.*

Proof. The right-to-left implication is easily checked. For the converse, suppose that $f \in \mathbf{R_3}(A \times A, A)$. Set $k = f(0,0)$. We have that: $R_{A\times A}(\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle)$, and $R_{A\times A}(\langle 0,0\rangle, \langle 1,0\rangle, \langle 2,0\rangle)$, and $R_{A\times A}(\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle)$. So there are apparently five choices for the four values $(f(0,0), f(0,1), f(1,0), f(1,1))$ namely: *(i)* (k,k,k,k); *(ii)* $(k,k,k,k+1)$; *(iii)* $(k,k,k+1,k+1)$; *(iv)* $(k,k+1,k,k+1)$; *(v)* $(k,k+1,k+1,k+1)$.

However, *(ii)* and *(v)* are impossible. We show this for *(v)*. Clearly *(v)* requires that $f(0,2) = k+2$ and, because $R_{A\times A}(\langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle)$, that $f(1,2) = k+1$. But then, as $R_{A\times A}(\langle 0,2\rangle, \langle 1,2\rangle, \langle 2,2\rangle)$, there is no possible value for $f(2,2)$.

We claim that for the other cases: *(i)* determines f to be $\langle m,n\rangle \mapsto k$; *(iii)* determines f to be $\langle m,n\rangle \mapsto m+k$; and *(iv)* determines f to be $\langle m,n\rangle \mapsto n+k$. We show this for *(iv)*. Clearly *(iv)* determines that $f(2,0) = k$ and that $f(2,1) = k+1$. Now a simple inductive argument shows, for all m, that $f(m,0) = k$ and $f(m,1) = k+1$. But then it is clear that $f(m,2) = k+2$, and another inductive argument shows that indeed $f(m,n) = n+k$. ⊠

It is now straightforward to show that, for example, the two distinct long-$\beta\eta$ normal forms, $\lambda p.\, \lambda l.\, p(p(l)(l))(l)$ and $\lambda p.\, \lambda l.\, p(p(l)(p(l)(l)))(l)$, of type $\top$, are interpreted as the same functional in $[\![\top]\!]$ (as are any two "trees" such that both leftmost branches have the same length, h say, and both rightmost branches have length k say). Thus we have shown that completeness cannot be reduced to completeness for the single type $(0 \to 0) \to 0 \to 0$.

Another direction in which one might hope to improve Corollary 4 would be to characterize the complete interpretations of Λ_X for arbitrary X. One would prefer a characterization that is both simple and useful (like Corollary 4), but unfortunately we do not have one. Here we content ourselves with showing that a most naïve attempt at a generalization of Corollary 4 fails. Specifically, define $\top_\alpha$ to be the type $(\alpha \to \alpha \to \alpha) \to \alpha \to \alpha$. We show that it is not necessarily the case that $[\![\cdot]\!] : \Lambda_X \to \mathcal{C}$ is complete when, for all $\alpha \in X$, for all $M^{\top_\alpha}$, $N^{\top_\alpha}$, it holds that $[\![M]\!] = [\![N]\!]$ implies $M =_{\beta\eta} N$. For a counterexample take $X = \{0, 0'\}$ and $\mathcal{C} = \mathbf{Set} \times \mathbf{Set}$. Define $[\![0]\!] = (\omega, \emptyset)$ and $[\![0']\!] = (\emptyset, \omega)$. By the completeness of Λ_X in $\mathbf{Set}$ we have that, for all $\alpha \in X$, for all $M^{\top_\alpha}$, $N^{\top_\alpha}$, it holds that $[\![M]\!] = [\![N]\!]$ implies $M =_{\beta\eta} N$. However, one sees that $[\![0 \to 0']\!]$ is interpreted as $(\emptyset, \mathbf{1})$ and so, for example, the two distinct terms (modulo $=_{\beta\eta}$) of $(0 \to 0') \to (0 \to 0') \to (0 \to 0')$ are interpreted as the same (unique) point of $[\![(0 \to 0') \to (0 \to 0') \to (0 \to 0')]\!]$. It follows that $[\![\cdot]\!]$ is not complete. We leave the finding of a useful characterization of complete interpretations of Λ_X as an open question. A related question is to find the simplest set of types to which $=_{\beta\eta}$ can be reduced in the manner of Proposition 2.

7 Discussion

It is clear that the work presented in this paper is heavily dependent on old results of Statman. In particular we use Theorem 3 of [15] (our Proposition 2) in a critical way, and our Theorem 2 is not too difficult a consequence of it. Further, we saw in Section 5 that Theorem 1 could also be derived as a fairly straightforward consequence of Statman's typical ambiguity theorem. However, although our main results follow without too much effort from Statman's work, the elegance and generality of our theorems makes them compelling semantic alternatives to Statman's syntactic results. We also hope that the present paper will have the effect of drawing attention to Statman's results, whose implications deserve to be better known.

Two departures from Statman's work are that we work with a calculus with unit and product types and that we allow more than one base type. The former difference is overcome using the characterization of $=_{\beta\eta}$ in terms of long $\beta\eta$-normal forms, which until quite recently was a field of active research (see, e.g., [1, 4, 7]). The latter difference turns out to be irrelevant in the case of Theorems 1 and 2 (as is shown by Proposition 1). In Section 6 we saw that this difference is non-trivial for the question of characterizing when an interpretation is complete.

It is interesting to compare our work with Statman's own semantic application of his syntactic results. In [17] he states his important *1-Section Theorem* giving necessary and sufficient conditions for an interpretation of $\Lambda^{\rightarrow}_{\{0\}}$ in a Henkin model to be complete. (See [12] for a detailed discussion and proof of the theorem.) The 1-Section Theorem is closely related to our Corollary 4, but it goes further, reducing completeness at the second-order type $\top$ to a property of elements of first-order types in a countable direct-product of the model. However, in doing so, the 1-Section Theorem makes essential use of the "well-pointedness" of Henkin models. There is a natural analogue of the 1-Section Theorem for well-pointed cartesian-closed categories, but not for general cartesian-closed categories. In this paper we have preferred not to consider results that apply only to well-pointed categories. After all, one of the benefits of the categorical setting is that non-well-pointed structures (such as closed-term categories) are handled alongside (the more set-theoretic) well-pointed structures in a uniform semantic framework. Note that our derivation of Theorem 3 from Theorem 1 made essential use of the applicability of our results to non-well-pointed categories.

One question is whether the results can be generalized to give completeness results for Λ_X augmented with typed constants. Categorically, one then considers CC-functors from the free cartesian closed category generated by a graph. Čubrić used Friedman's techniques to show that there is a faithful CC-functor from any such free CCC to **Set** [4]. Unfortunately, our proofs do not extend in this way, as Proposition 2 fails once constants are added to the syntax.

Another interesting question is whether the purely categorical formulations of Theorems 1 and 2 extend to other kinds of categories with structure. It seems likely that both results will generalize to bicartesian closed categories. The main obstacle in proving such a generalization is to get a good handle on equality in the internal language. It is already difficult to generalize long-$\beta\eta$ normal forms (although see [6] for progress on this question), let alone the deep syntactic results of Statman. On the other hand, for recursion theoretic reasons, it is clear that our results do not generalize to cartesian-closed categories with a natural numbers object.

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